

NUMBER OF NODAL DOMAINS OF EIGENFUNCTIONS ON NON-POSITIVELY CURVED SURFACES WITH CONCAVE BOUNDARY

JUNEHYUK JUNG AND STEVE ZELDITCH

ABSTRACT. It is an open problem in general to prove that there exists a sequence of Δ_g -eigenfunctions φ_{j_k} on a Riemannian manifold (M, g) for which the number $N(\varphi_{j_k})$ of nodal domains tends to infinity with the eigenvalue. Our main result is that $N(\varphi_{j_k}) \rightarrow \infty$ along a subsequence of eigenvalues of density 1 if the (M, g) is a non-positively curved surface with concave boundary, i.e. a generalized Sinai or Lorentz billiard. Unlike the recent closely related work of Ghosh-Reznikov-Sarnak and of the authors on the nodal domain counting problem, the surfaces need not have any symmetries.

1. INTRODUCTION

Let (M, g) be a surface with non-empty smooth boundary $\partial M \neq \emptyset$. We consider the eigenvalue problem,

$$\begin{cases} -\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda, \\ B\varphi_\lambda = 0 \text{ on } \partial M \end{cases},$$

where B is the boundary operator, e.g. $B\varphi = \varphi|_{\partial M}$ in the Dirichlet case or $B\varphi = \partial_\nu \varphi|_{\partial M}$ in the Neumann case. We denote by $\{\varphi_j\}$ an orthonormal basis of eigenfunctions, $\langle \varphi_j, \varphi_k \rangle = \delta_{jk}$, with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ counted with multiplicity. The inner product is defined by $\langle f, g \rangle = \int_M f \bar{g} dA$ where dA is the area form of g . We denote the nodal line of φ_λ by

$$Z_{\varphi_\lambda} = \{x : \varphi_\lambda(x) = 0\}.$$

We also denote by $N(\varphi_\lambda)$ the number of nodal domains of φ_λ , i.e. the number of connected components Ω_j of

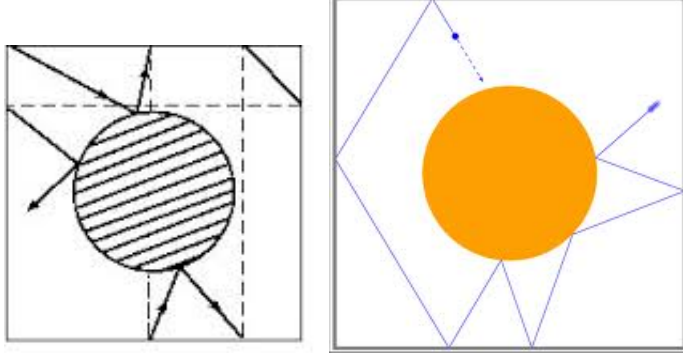
$$M \setminus (Z_{\varphi_\lambda} \cup \partial M) = \bigcup_{j=1}^{N(\varphi_\lambda)} \Omega_j.$$

The connected components are called the nodal domains of φ_λ . We further denote by

$$\Sigma_{\varphi_\lambda} = \{x \in Z_{\varphi_\lambda} : d\varphi_\lambda(x) = 0\}$$

the singular set of φ_λ . The main result of this article, developing the method of [JZ13], gives a rather general sufficient condition on surface M with non-empty boundary $\partial M \neq \emptyset$ and ergodic billiard flow under which the number of nodal domains tends to infinity along a full density subsequence (i.e. of

‘almost all’ eigenfunctions of any orthonormal basis). A significant gain in the billiard case is that we do not require the surface (or eigenfunctions) to have a symmetry.



We first state the result for the special case of “Sinai-Lorentz” billiards, i.e. non-positively curved surfaces with concave boundary. We denote the scalar curvature of (M, g) by K .

Theorem 1.1. *Let (M, g) be a non-positively curved surface $K \leq 0$ with non-empty smooth concave boundary ∂M . Let $\{\varphi_j\}$ be an orthonormal eigenbasis of Dirichlet (resp. Neumann) eigenfunctions. Then there exists a subsequence $A \subset \mathbb{N}$ of density one so that*

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} N(\varphi_j) = \infty.$$

By a surface of non-positive curvature with concave boundary, we mean a non-positively curved surface (M, g) ,

$$M = X \setminus \bigcup_{j=1}^r \mathcal{O}_j, \quad (1.1)$$

obtained by removing a finite union $\mathcal{O} := \bigcup_{j=1}^r \mathcal{O}_j$ of embedded nonintersecting geodesically convex domains (or ‘obstacles’) \mathcal{O}_j from a closed non-positively curved surface (X, g) without boundary. It is proved in [Sin70, CS87, BSC90] that the billiard flow of a Sinai-Lorentz billiard is ergodic.

We give in fact a more general condition, which requires some further notation and terminology.

Definition 1.2. *The Cauchy data of Neumann (resp. Dirichlet) eigenfunctions of (M, g) is defined by*

$$\begin{cases} \varphi_j^b = \varphi_j|_{\partial M}, & \text{Neumann boundary conditions,} \\ \varphi_j^b = \lambda_j^{-1} \partial_\nu \varphi_j|_{\partial M}, & \text{Dirichlet boundary conditions,} \end{cases}$$

The more general result is:

Theorem 1.3. *Let (M, g) be a surface with non-empty smooth boundary ∂M . Let $\{\varphi_j\}$ be an orthonormal eigenbasis of Dirichlet (resp. Neumann) eigenfunctions. Assume that (M, g) satisfies the following conditions:*

- (i) *The billiard flow G^t is ergodic on S^*M with respect to Liouville measure;*
- (ii) *The sup norms of the Cauchy data*

$$(\varphi_j|_{\partial M}, \lambda_j^{-1} \partial_\nu \varphi_j|_{\partial M})$$

are $o(\lambda_j^{\frac{1}{2}})$ as $\lambda_j \rightarrow \infty$.

Then there exists a subsequence $A \subset \mathbb{N}$ of density one so that

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} N(\varphi_j) = \infty.$$

Theorem 1.1 follows from Theorem 1.3 combined with some new results of the second author with C. Sogge. The ergodicity condition (i) is known to be satisfied for a non-positively curved surface with concave boundary [KSS89] (see §2). Moreover, in [SZ14] the second condition (ii) is proved for such surfaces, among many other cases. Using the Melrose-Taylor diffractive parametrix on manifolds with concave boundary, the following is proved:

Theorem 1.4. [SZ14] *Let (M, g) be a Riemannian manifold of dimension n with geodesically concave boundary. Suppose that there exist no boundary self-focal points $q \in \partial M$. Then the sup-norms of Cauchy data of Dirichlet, resp. Neumann, eigenfunctions are $o(\lambda_j^{\frac{n-1}{2}})$.*

The term *boundary self-focal point* is a dynamical condition on the billiard flow Φ^t of $(M, g, \partial M)$, i.e. the geodesic flow in the interior with elastic reflection on ∂M . See §2 for background. We denote the broken exponential map by $\exp_x \xi = \pi \Phi^1(x, \xi)$ (see [Hör90] (Chapter XXIV) or [MT85] for background.) Given any $x \in \bar{M}$, we denote by \mathcal{L}_x the set of loop directions at x ,

$$\mathcal{L}_x = \{\xi \in S_x^*M : \exists T : \exp_x T\xi = x\}. \quad (1.2)$$

Definition 1.5. *We say that q is a boundary self-focal point if $|\mathcal{L}_q| > 0$ where $|\cdot|_q$ denotes the surface measure on $S_{q, \text{in}}^*M$, the set of inward pointing unit tangent vectors determined by the metric g_q . Equivalently, the measure of the set of $\xi \in B_q^* \partial M$ of the co-ball bundle $B_q^* \partial M$ of the boundary at q for which $\pi \beta(q, \xi) = (q, \eta)$ for some $\eta \in B_q^* \partial M$ has measure zero. Here, β is the billiard map (2.3).*

Billiards on non-positively curved billiards on (M, g) with concave boundary never have self-focal points. Self-focal points q are necessarily self-conjugate, i.e. there exists a broken Jacobi field along a geodesic billiard loop at q vanishing at both endpoints. But as we review in §2, non-positively

curved dispersive billiards as above do not have conjugate points. Thus, Theorem 1.4 holds and so all of the hypotheses of Theorem 1.3 are satisfied by the surfaces of Theorem 1.1.

Theorem 1.4 is the analogue for Cauchy data of the sup norm results on manifolds without boundary proved in [SZ02, SZ13]. The proof is too lengthy to be included here and will be published elsewhere.

1.1. Outline of the proof of Theorem 1.3. We plan to deduce Theorem 1.3 from Theorem 1.4 and from a variety of additional analytical and topological arguments. The principal analytical results, besides Theorem [SZ02], are the quantum ergodic restriction theorems of [HZ04, CTZ13]. The overall argument follows the strategy of [JZ13], and is based on counting the zeros of the Cauchy data φ_j^b of an ergodic sequence of eigenfunctions on ∂M and relating them to numbers of nodal domains using the Euler inequality for embedded graphs.

We now sketch the proof of Theorem 1.3. The analytical part of the proof is the following result about boundary nodal points:

Theorem 1.6. *Let (M, g) be a Riemannian surface which satisfies the assumptions of Theorem 1.3. Then for any given orthonormal eigenbasis of Neumann eigenfunctions $\{\varphi_j\}$, there exists a subsequence $A \subset \mathbb{N}$ of density one such that*

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} \#Z_{\varphi_j} \cap \partial M = \infty.$$

Furthermore, there are an infinite number of zeros where $\varphi_j|_{\partial M}$ changes sign. For Dirichlet eigenbasis $\{\varphi_j\}$, there exists a subsequence $A \subset \mathbb{N}$ of density one such that

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} \#\Sigma_{\varphi_j} \cap \partial M = \infty.$$

Given Theorem 1.6, the remainder of the proof of Theorem 1.1 is topological. As in [JZ13], we use an Euler characteristic argument for embedded graphs, Theorem 6.3, to obtain a lower bound on the number of nodal domains from the lower bound on the number of boundary zeros. Heuristically, the nodal lines which touch the boundary transversally must intersect at two points of the boundary and trap nodal domains. This is not literally correct when the genus is > 0 but the genus correction is bounded and does not affect the growth rate of the number of nodal domains. The role of the boundary in [GRS13] and [JZ13] was played by the fixed point set of an anti-holomorphic involution. No symmetry assumption is necessary in the boundary case, because a nodal segment which has an end point at ∂M must terminate in ∂M .

1.2. Outline of the proof of Theorem 1.6. The key step in proving Theorem 1.6 is the following

Proposition 1.7. *Let (M, g) be a Riemannian surface with smooth boundary and ergodic billiards. Then exists a subsequence of density one of the Neumann eigenfunctions such that, for any fixed arc (or interval) $\beta \subset \partial M$,*

$$\int_{\beta} |\varphi_j| ds > \left| \int_{\beta} \varphi_j ds \right|.$$

Moreover, there exists a density one subsequence of Dirichlet eigenfunctions such that for any fixed arc $\beta \subset \partial M$,

$$\int_{\beta} |\partial_{\nu} \varphi_j| ds > \left| \int_{\beta} \partial_{\nu} \varphi_j ds \right|.$$

That is,

$$\int_{\beta} |\varphi_j^b| ds > \left| \int_{\beta} \varphi_j^b ds \right|. \quad (1.3)$$

The Proposition clearly implies that the Neumann eigenfunctions φ_{λ} must have a sign-changing zero on any arc β for a density one subsequence of eigenfunctions. Similarly for normal derivatives of Dirichlet eigenfunctions. Theorem 1.6 is a direct consequence of Proposition 1.7. By an ‘arc’ or interval of ∂M we simply mean the image of an interval under a parametrization.

To prove Proposition 1.7 we combine three results, two of which are proved elsewhere and one which we prove here. The first is the quantum ergodic restriction theorem (1.4) discussed above. The second ingredient is the following “Kuznecov sum formula”, extending the result of [Zel92] to manifolds with boundary. It is an immediate consequence of Theorem 1 of [HHHZ13]:

Theorem 1.8 ([HHHZ13]). *Let (M, g) be a Riemannian surface with smooth boundary ∂M . Let $\{\varphi_j\}$ be an orthonormal basis of Neumann eigenfunctions. For any given fixed $f \in C_0^{\infty}(\partial M)$,*

$$\sum_{\lambda_j < \lambda} \left| \int_{\partial M} f \varphi_j ds \right|^2 = \left(\frac{2}{\pi} \int_{\partial M} f^2 ds \right) \lambda + o(\lambda).$$

Let $\{\varphi_j\}$ be an orthonormal basis of Dirichlet eigenfunctions. For any given fixed $f \in C_0^{\infty}(\partial M)$,

$$\sum_{\lambda_j < \lambda} \left| \lambda_j^{-1} \int_{\partial M} f \partial_{\nu} \varphi_j ds \right|^2 = \left(\frac{2}{\pi} \int_{\partial M} f^2 ds \right) \lambda + o(\lambda).$$

We prove Theorem 1.8 in §5. We view Theorem 1.8 as an asymptotic mean formula for a sequence of probability measures, stating that on average, $\left| \int_{\partial M} f \varphi_j^b ds \right|^2$ is of order λ_j^{-1} . An application of Chebychev’s inequality then gives

Corollary 1.9. *There exists a constant $c = c_f > 0$ depending only on f such that, for each $M > 0$ there exists a subsequence of proportion $\geq 1 - \frac{c}{M}$*

of the $\{\varphi_j\}$ for which

$$\left| \int_{\partial M} f \varphi_j^b ds \right| \leq M \lambda_j^{-\frac{1}{2}}$$

The next ingredient is the quantum ergodicity of Cauchy data of ergodic sequences of eigenfunctions along the boundary. In [HZ04, CTZ13] (see also [Bur05]) it is proved that if the geodesic billiard flow of a Riemannian manifold (M, g) is ergodic, then there exists a subsequence $A \subset \mathbb{N}$ of density one so that, for any $f \in C(M)$,

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} \int_{\partial M} f |\varphi_j^b|^2 ds = \omega_B(f), \quad (1.4)$$

where $\omega_B(f)$ is a ‘limit state’ (i.e. a positive measure viewed as a linear functional on $C(B^*\partial M)$) which depends on the boundary condition B. The limit state is defined in §3 and the result is stated in Theorem 3.1. We refer to the density one sequence (1.4) as a “boundary ergodic sequence” of eigenfunctions.

We now put together Theorem 1.4, Theorem 1.8, and (1.4) to give an outline of the proof Proposition 1.7. For simplicity we assume that the boundary conditions are Neumann. From the third assumption in Theorem 1.1, we have

$$\|\varphi_j|_{\partial M}\|_{L^\infty} \leq \lambda_j^{1/2} o(1).$$

We then prove that for any $M > 0$ there exists a subsequence of density $\geq 1 - \frac{1}{M}$ so that for any arc $\beta \subset H$,

$$\left| \int_{\beta} \varphi_j ds \right| \leq C \lambda_j^{-1/2} M$$

and there exists a subsequence of density one for which

$$\int_{\beta} |\varphi_j| ds \geq \|\varphi_j\|_{C^0(\beta)}^{-1} \|\varphi_j\|_{L^2(\beta)}^2 \geq C \lambda_j^{-1/2} \frac{1}{o(1)},$$

This is a contradiction if φ_j has no sign change on β .

It follows that for any $M > 0$ there exists a subsequence of density $\geq 1 - \frac{1}{M}$ for which

$$\int_{\beta} |\varphi_j| ds > \left| \int_{\beta} \varphi_j ds \right|.$$

This implies the existence of a subsequence of density one with this property.

1.3. Related and future work. For the sake of completeness, we recall that in [JZ13] it is proved that $N(\varphi_\lambda)$ tends to infinity along a density one sequence of even or odd eigenfunctions on a Riemannian surfaces (M, g, σ) of negative curvature with an isometric orientation reversing involution σ with separating fixed point set $\text{Fix}(\sigma)$. In [JZ13], the assumption of the existence of σ is necessary in order to ensure that there are at most two intersections between $\text{Fix}(\sigma)$ and a closed nodal curve, which allows one to relate the number of nodal domains and number of intersections.

We also conjecture that Theorem 1.3 generalizes to all hyperbolic billiards, such as planar billiard tables with concave walls and corners, and also to the Bunimovich stadium (with a continuous tangent line). The only part of the proof which is not contained in this article is the validity of the conclusion of Theorem 1.4 on such manifolds with corners. The corners are serious complications to the proof of Theorem 1.4 since sequences of eigenfunctions might be exceptionally large there. However for the purposes of this article, sup norm estimates on these manifolds are only needed away from the corners, and it may be possible to prove the necessary bounds without proving all of Theorem 1.4 in the cornered case.

We also conjecture that Theorem 1.4 should be true for general Riemannian manifolds with smooth boundary with no boundary self-focal points. This is work in progress of the second author with C. Sogge. If we can generalize Theorem 1.4 to any smooth boundary with no boundary self-focal points, then the condition (ii) in Theorem 1.3 can be replaced by the purely dynamical condition, [(ii)'] There does not exist a boundary *self-focal* point $q \in \partial M$ for the billiard flow, and one would have purely dynamical conditions (i)-(ii)' ensuring growth of numbers of nodal domains (as in the special case of Theorem 1.1). One may further ask if ergodicity itself prohibits existence of self-focal points on the boundary, at least in the real analytic case. As observed in [SZ02], real analytic surfaces without boundary and ergodic geodesic flow cannot have self-focal points p because the flowout of S_p^*M under the geodesic flow is an invariant Lagrangian torus enclosing a positive measure invariant set. But in the boundary case, the flowout of the inward pointing $S_{p,\text{in}}^*M$ at $p \in \partial M$ might not bound an invariant set.

The intersection points $Z_{\varphi_\lambda} \cap \partial M$ in the Neumann case are the points where the nodal set touches the boundary in the sense of [TZ09]. That article shows that if ∂M is piecewise real analytic, then the number of intersection points is $\leq C_M \lambda$ for some constant depending only on the domain. In theorem 1.6, we only show that the number of intersection points tends to infinity in our setting. It would be very interesting to have a quantitative lower bound. For Sinai-Lorentz billiards we conjecture that the sup norms of the Cauchy data are of order $O(\frac{\lambda_j^{\frac{n-1}{2}}}{\sqrt{\log \lambda_j}})$. This would be a key step in producing zeros in intervals of the boundary of lengths $\frac{1}{\sqrt{\log \lambda_j}}$, thereby producing logarithmic lower bounds on numbers of nodal domains.

1.4. Acknowledgements. This paper makes use of recent joint work of the second author with several collaborators, which in part was motivated by the applications to nodal sets. Theorem 1.8 is recent joint work with X. Han, A. Hassell and H. Hezari [HHHZ13]. It also uses calculations in recent work [HZ12] with H. Hezari. As mentioned above, Theorem 1.4 is joint work with C. D. Sogge [SZ14]. The boundary quantum ergodicity theorem and boundary local Weyl law is joint work with A. Hassell [HZ04]

and with H. Christianson and J. Toth [CTZ13]. We would also like to thank N. Simanyi for helpful correspondence regarding [KSS89] and L. Stoyanov for correspondence on billiard problems.

The first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP)(No. 2013042157). The first author was also partially supported by NSF grant DMS-1128155 and by TJ Park Post-doc Fellowship funded by POSCO TJ Park Foundation. Research of the second author was partially supported by NSF grant DMS-1206527.

2. BILLIARDS ON NEGATIVELY CURVED SURFACES EXTERIOR TO CONVEX OBSTACLES

Before discussing billiards on these surfaces, we introduce some general notation and background.

2.1. Billiard map. We denote by

$$\Phi^t : S^*M \rightarrow S^*M \quad (2.1)$$

the billiard (or broken geodesic) flow on S^*M . The trajectory $\Phi^t(x, \xi)$ consists of geodesic motion between impacts with the boundary, with the usual reflection law at the boundary.

The billiard map β is defined on the unit ball bundle $B^*\partial M$ of ∂M as follows: given $(y, \eta) \in B^*\partial M$, i.e. with $|\eta| < 1$, we let $(y, \zeta) \in S^*M$ be the unique inward-pointing unit covector at y which projects to (y, η) under the map $T_{\partial M}^*\bar{M} \rightarrow T^*\partial M$. Then we follow the geodesic

$$(y, \eta) \in B^*(\partial M) \rightarrow \Phi^t(q(y), \zeta(y, \eta)) \quad (2.2)$$

until the projected billiard orbit first intersects the boundary again; let $y' \in \partial M$ denote this first intersection. We denote the inward unit normal vector at y' by $\nu_{y'}$, and let $\zeta' = \zeta + 2(\zeta \cdot \nu_{y'})\nu_{y'}$ be the direction of the geodesic after elastic reflection at y' . Also, let η' be the projection of η' to $T_{y'}^*\partial M$. Then

$$\beta(y, \eta) := (y', \eta'). \quad (2.3)$$

The directions tangent to ∂M cause singularities (discontinuities) in β , and the billiard map is not a priori well-defined on initial directions which are tangent to ∂M , i.e. $(y, \eta) \in S^*\partial M$ with $|\eta| = 1$. To obtain a smooth dynamical system, one often removes the tangential directions $S^*\partial M$. Billiard trajectories starting on ∂M in non-tangential directions may become tangential at some future intersection and one often punctures out all trajectories which in the past or future become tangential. That is, one defines $B_q^0 = B_q^*\partial M \setminus S_q^*\partial M$ to be the projections of non-tangential directions and define $\mathcal{R}^1 \subset B_q^0$ to be $\beta^{-1}(B_q^0)$ and, more generally, $\mathcal{R}^{k+1} = \beta^{-1}(\mathcal{R}^k)$ for natural numbers k . Thus \mathcal{R}^k consists of the points where β^k is well defined and maps to B_q^0 . Similarly one defines $\mathcal{R}^{-1} = \beta_-^{-1}B_q^0$ and $\mathcal{R}^{-k-1} = \beta_-^{-1}(\mathcal{R}^{-k})$.

Clearly $\mathcal{R}^1 \supset \mathcal{R}^2 \supset \dots$, $\mathcal{R}^{-1} \supset \mathcal{R}^{-2} \supset \dots$ and it is shown in [CFS82] that each \mathcal{R}^k has full measure. Then β is a symplectic diffeomorphism of

$$\mathcal{R}^\infty = \bigcap_k \mathcal{R}^k.$$

However, in this article we define β at tangent vectors to ∂M so that the billiard trajectory is simply a geodesic of (X, g) which hits ∂M tangentially. The billiard map is then defined on all of $B^*\partial M$ but is discontinuous along the set of tangential directions.

2.2. Ergodicity of the billiard map for non-positively curved surfaces with concave boundary. We need the following result of Kramli-Simonyi-Szasz [KSS89].

Proposition 2.1 ([KSS89]). *Billiards on a non-positively curved surface with concave boundary are ergodic.*

Since the result is not stated this way in [KSS89], we briefly review the proof. The main result of the [KSS89] is a proof of the “Fundamental Theorem for Dispersing Billiards” (Theorem 5.1). The “dispersing billiards” condition implies the existence almost everywhere of stable/unstable foliations for the billiard map and also indicates some of its quantitative properties. In section 6 of [KSS89] the authors show how ergodicity of the billiards follows from their Theorem 5.1 by the so-called Hopf-Sinai argument.

The surfaces in [KSS89] are assumed to be the exterior (1.1) of a finite union of convex obstacles (i.e. the boundary curves have strictly positive geodesic curvature from the inside). They are more general than the non-positively curved surfaces assumed here. The billiards are only assumed to satisfy ‘Vetier’s conditions’, which are conditions implying that ‘no focal points arise’. The conditions are stated precisely in Condition 1.2-1.4 in [KSS89]. Condition (1.2) is that the distance between obstacles is bounded below by some $\tau_{\min} > 0$, which is obvious for a compact surface when the obstacles do not intersect. Condition (1.3) is that there exists τ_{\max} so that any geodesic must intersect ∂M in time $\leq \tau_{\max}$. This condition can be removed if the curvature is strictly negative. Condition (1.4) is a curvature condition which is satisfied as long as $K \leq 0$. In this case, Condition (1.3) becomes irrelevant. In fact, $K \leq 0$ alone implies the fundamental theorem and ergodicity of the billiard flow.

For $(x, \xi) \in S_x^*M$, the stable (resp. unstable) fiber $H^{(s)}(x, \xi)$ (resp. $H^{(u)}(x, \xi)$) through x is the set of $(y, \eta) \in S^*M$ so that

$$\lim_{t \rightarrow +\infty} d(\Phi^t(y, \eta), \Phi^t(x, \xi)) = 0$$

(resp. $t \rightarrow -\infty$). The stable leaf through x is $\bigcup_{t \in \mathbb{R}} \Phi^t(H^{(s)}(x))$. Similarly for the unstable leaf. Vetier proved that under the conditions above, there exist stable and unstable fibers through almost $(x, \xi) \in S^*M$ which are C^1

curves. It follows that through almost every $(q, \eta) \in B^*\partial M$ there exist stable/unstable leaves for the billiard map, which are C^1 curves. In particular, this is the case for non-positively curved surfaces with concave boundary.

2.3. Absence of self-focal points non-positively curved surfaces with concave boundary. The follow Lemma lets us use Theorem 1.4 in Theorem 1.3.

Lemma 2.2. *There do not exist partial self-focal points on a non-positively curved surfaces with concave boundary.*

To prove this, we consider broken Jacobi fields for the billiard flow and begin with some background from [Woj94, ZL07, Bia13]. A (normal) Jacobi field is an orthogonal vector field $J(t)$ along a billiard trajectory γ with transversal reflections at ∂M , which satisfies the Jacobi equation $\frac{D^2}{dt^2}J + K(\gamma(t))J = 0$ away from the elastic impacts, and which is reflected by the law

$$\begin{pmatrix} J \\ J' \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ \frac{2K(s)}{\sin \varphi(s)} & -1 \end{pmatrix} \begin{pmatrix} J \\ J' \end{pmatrix} = \begin{pmatrix} -J \\ \frac{2K(s)}{\sin \varphi(s)}J - J' \end{pmatrix}$$

at the reflection point. Here φ is the angle that $\gamma'(t)$ makes with the boundary at the impact time. Recalling that a Jacobi field is the variation vector field $J(t) = \frac{\partial}{\partial \varepsilon} \gamma_\varepsilon(t)$ of a 1-parameter family of billiard trajectories, we see that the reflection law is the derivative in ε of the reflection law for the curves γ_ε .

Proof. Assume first that Y is a simply connected non-positively curved surface, and that $\mathcal{O}_1, \dots, \mathcal{O}_m \subset Y$ are disjoint obstacles. Fix a point $q \in \partial \mathcal{O}_1$ and consider billiard trajectories of (q, η) on $Y \setminus \bigcup_{j=1}^m \mathcal{O}_j$. For a given billiard trajectory of (q, η) , we correspond a sequence $\{a_j(\eta)\}_{j \geq 0}$ with $1 \leq a_j(\eta) \leq m$ such that j -th impact occurs on the boundary of \mathcal{O}_{a_j} (we assume that $a_0(\eta) = 1$.)

For a given sequence $B = \{b_j\}_{0 \leq j \leq M}$ with $1 \leq b_j \leq m$, let

$$S_B := \{\eta \mid a_j(\eta) = b_j, j = 0, \dots, M\}.$$

Let $q_j(\eta) \in \partial \mathcal{O}_{a_j(\eta)}$ be the j -th impact point. Since Y is simply connected, the length of billiard trajectory from q to $q_M(\eta)$ is bounded from above by some constant for $\eta \in S_B$. Therefore, if there are infinitely many $\eta \in S_B$ such that $q_M(\eta) = q'$ with same q' , then q and q' are conjugate. However, this is impossible, since the norm of the Jacobi field is monotonically increasing between impacts and only grows at an impact. Hence along any billiard trajectory from any $q \in \partial M$, it cannot vanish for $t > 0$.

Now let \tilde{X} be the universal covering of X and let $\tilde{\mathcal{O}}_j$ for $j = 1, 2, \dots$ be the lift of obstacles on X . Fix $q \in \partial M$. For each $(q, \eta) \in S^*M$ whose billiard trajectory is a loop, we correspond a finite length sequence $\{a_j(\eta)\}_{j \leq T}$ as above, where $T > 0$ is the first index such that $q_T(\eta)$ is a preimage of q

on \tilde{X} . Then from above, we infer that there are finitely many η such that $\{a_j(\eta)\}_{j \leq T} = B$ for any given B . Since the set of B is countable, this proves that there are at most countably many η , for which billiard trajectory of (q, η) is a loop. \square

Remark: The Lemma can also be extracted from the articles [Sto99, Sto89] of L. Stoyanov.

3. BOUNDARY QUANTUM ERGODIC RESTRICTION THEOREMS

In this section, we briefly review the statement of the quantum ergodic restriction theorem to the boundary of [HZ04]. Roughly speaking, the results says that if the billiard ball map β on $B^*\partial M$ is ergodic, then the boundary values (Cauchy data) u_j^b of eigenfunctions are quantum ergodic. We define

$$\gamma(q) = \sqrt{1 - |\eta|^2}, \quad q = (y, \eta), \quad (3.1)$$

and put

B	$B\varphi_\lambda$	φ_λ^b	$d\mu_B$
Dirichlet	$u _Y$	$\lambda^{-1}\partial_\nu\varphi_\lambda _Y$	$\gamma(q)d\sigma$
Neumann	$\partial_\nu\varphi_\lambda _Y$	$\varphi_\lambda _Y$	$\gamma(q)^{-1}d\sigma$

Table 1: Boundary Values

The third column consists of the non-zero part of the Cauchy data (Definition 1.2) for eigenfunctions satisfying the associated boundary condition. The measures in the right column are the so-called quantum limits of the Cauchy data.

Theorem 3.1. [HZ04, CTZ13, Bur05] *Let M be a compact manifold with boundary and with ergodic billiard map. Let $\{\varphi_j^b\}$ be the boundary values of the eigenfunctions $\{\varphi_j\}$ of Δ_B on $L^2(M)$ in the sense of the table above. Let A_h be a semiclassical operator of order zero on ∂M . Then there is a subset S of the positive integers, of density one, such that*

$$\lim_{j \rightarrow \infty, j \in S} \langle A_{h_j} \varphi_j^b, \varphi_j^b \rangle = \omega_B(A), \quad (3.2)$$

where $h_j = \lambda_j^{-1}$ and ω_B is the classical state on the table above.

We only apply the result to ‘multiplication operators’ by $f \in C(\partial M)$ in this article, in which case it takes the form (1.4).

The main idea of Theorem 3.1 is that the Cauchy data of the eigenfunctions provide a quantum analogue of a cross-section for the billiard flow, just as the inward pointing unit tangent vectors $S_{\text{in}, \partial M}^* M$ along the boundary provide a classical cross section. The interior quantum ergodicity of

eigenfunctions thus implies boundary quantum ergodicity. In [CTZ13], this results was generalized to any smooth hypersurface of M . For Theorem 1.3 we only need the case where the hypersurface is the boundary.

4. RESTRICTION OF THE WAVE GROUP OF A SINAI BILLIARD TO THE BOUNDARY

In this section, we prepare for the proof of Theorem 1.8 by considering the wave front set of the Cauchy data of the Dirichlet, resp. Neumann, wave kernel along the boundary. We also recall the boundary local Weyl law.

The billiard flow arises in spectral problems because the singularities of the fundamental solution of the wave equation propagate along billiard trajectories. The billiard flow relevant to our problem is thus determined by propagation of singularities for the wave equation on a non-positively curved surface (1.1) in the exterior of a finite union of disjoint convex obstacles. In this case, the singularities which intersect the boundary are known as grazing rays. It was proved independently by R. B. Melrose and M. E. Taylor [Mel75, Tay76] that a singularity propagating along a grazing ray simply continues along the same geodesic of the ambient space when it touches the boundary. Consequently, the dynamical billiard flow of the previous section coincides with the propagation of singularities.

The Melrose-Taylor diffractive parametrix is used to determine the precise singularities of solutions of wave equations near grazing rays, i.e. billiard trajectories intersecting ∂M tangentially. In [SZ14], the diffractive parametrix of Melrose-Taylor is used to prove Theorem 1.4. The remaining properties of the wave kernel needed for Theorem 1.3 are more elementary and do not require the diffractive parametrix.

4.1. Wave front set of the wave group. We denote by

$$E_B(t) = \cos(t\sqrt{-\Delta_B}), \quad \text{resp.} \quad S_B(t) = \frac{\sin(t\sqrt{-\Delta_B})}{\sqrt{-\Delta_B}} \quad (4.1)$$

the even (resp. odd) wave operators (M, g) with boundary conditions B . The wave group $E_B(t)$ is the solution operator of the mixed problem

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)E_B(t, x, y) = 0, \\ E_B(0, x, y) = \delta_x(y), \quad \frac{\partial}{\partial t}E_B(0, x, y) = 0, \quad x, y \in M; \\ BE_B(t, x, y) = 0, \quad x \in \partial M \end{cases}$$

The wave front sets of $E_B(t, x, y)$ and $S_B(t, x, y)$ are determined by the propagation of singularities theorem of [MS78] for the mixed Cauchy Dirichlet (or Cauchy Neumann) problem for the wave equation. We from [Hör90] (Vol. III, Theorem 23.1.4 and Vol. IV, Proposition 29.3.2) that

$$WF(E_B(t, x, y)) \subset \bigcup_{\pm} \Lambda_{\pm}, \quad (4.2)$$

where $\Lambda_{\pm} = \{(t, \tau, x, \xi, y, \eta) : (x, \xi) = \Phi^t(y, \eta), \tau = \pm|\eta|_y\} \subset T^*(\mathbb{R} \times \Omega \times \Omega)$ is the graph of the generalized (broken) geodesic flow, i.e. the billiard flow Φ^t . The same is true for $WF(S_B)$. As mentioned above, the broken geodesics in the setting of (1.1) are simply the geodesics of the ambient negatively curved surface, with the equal angle reflections at the boundary; tangential rays simply continue without change at the impact.

4.2. Restriction of wave kernels to the boundary. A key object in the proof of Theorem 1.4 is the analysis of the restriction of the Schwartz kernel $E_B(t, x, y)$ of $\cos t\sqrt{\Delta_B}$ to $\mathbb{R} \times \partial M \times \partial M$ and further to $\mathbb{R} \times \Delta_{\partial M \times \partial M}$, where $\Delta_{\partial M \times \partial M}$ is the diagonal of $\partial M \times \partial M$. We denote by dq the surface measure on the boundary ∂M , and by $ru = u|_{\partial M}$ the trace operator. We denote by $E_B^b(t, q', q) \in \mathcal{D}'(\mathbb{R} \times \partial M \times \partial M)$ the following boundary traces of the Schwartz kernel $E_B(t, x, y)$ defined in (4.1):

$$E_B^b(t, q', q) = \begin{cases} r_{q'} r_q \partial_{\nu_{q'}} \partial_{\nu_q} E_D(t, q', q), & \text{Dirichlet} \\ r_{q'} r_q E_N(t, q', q), & \text{Neumann} \end{cases} \quad (4.3)$$

The subscripts q', q refer to the variable involved in the differentiating or restricting. Henceforth we use the notation γ_q^B for the boundary trace. Thus, $\gamma_q^B = r_q$ in the Neumann case and $\gamma_q^B = r_q \partial_{\nu_q}$ in the Dirichlet case.

The sup norm bounds of Theorem 1.4 are derived in [SZ14] from an analysis of the singularities of the Cauchy data of the wave kernel on the diagonal of the boundary,

$$E_B^b(t, q, q) = \sum_{j=1}^{\infty} \cos(t\lambda_j) |\varphi_j^b(q)|^2. \quad (4.4)$$

The Kuznecov asymptotics of Theorem 1.8 are proved by studying the off-diagonal integrals,

$$\int_{\partial M} \int_{\partial M} E_B^b(t, q, q') ds(q) ds(q') = \sum_{j=1}^{\infty} \cos(t\lambda_j) \left| \int_{\partial M} \varphi_j^b(q) ds(q) \right|^2. \quad (4.5)$$

The contribution of the tangential directions to the singularities of (4.5) is negligible, and that is why the diffractive parametrix is not needed.

4.3. Wave front set of the restricted wave kernel. The first and simplest piece of information is the wave front set of (4.4). It follows from (4.2) and from standard results on pullbacks of wave front sets under maps, the wave front set of $E_B^b(t, q, q')$ consists of co-directions of broken trajectories which begin and end on ∂M . That is,

$$\begin{aligned} WF(\gamma_q^B \gamma_{q'}^B E(t, q, q')) &\subset \{(t, \tau, q, \eta, q', \eta') \in B^* \partial M \times B^* \partial M : \\ &[\Phi^t(q, \xi(q, \eta))]^T = (q', \eta'), \tau = -|\xi|\}. \end{aligned} \quad (4.6)$$

Here, the superscript T denotes the tangential projection to $B^*\partial M$. We refer to Section 2 of [HZ12] for an extensive discussion. It follows from (4.6) that

$$\begin{aligned} WF(\gamma_q^B \gamma_{q'}^B E(t, q, q)) &\subset \{(t, \tau, q, \eta, q, \eta') \in B_q^* \partial M \times B_q^* \partial M : \\ &[G^t(q, \xi(q, \eta))]^T = (q, \eta'), \quad \tau = -|\xi(q, \eta)|\}. \end{aligned} \quad (4.7)$$

Thus, for $t \neq 0$, the singularities of the boundary trace $\gamma_q^B \gamma_{q'}^B E(t, q, q)$ at $q \in \partial M$ to broken bicharacteristic loops based at q in \overline{M} . When $t = 0$ all inward pointing co-directions belong to the wave front set.

4.4. Local Weyl law. The boundary local Weyl law gives an asymptotic formula for the spectral averages of the expected value of an observable A_h relative to boundary traces of eigenfunctions. The relevant algebra of observables in our setting as in [HZ04] is the algebra $\Psi_h^0(\partial M)$ of zeroth order semiclassical pseudodifferential operators on ∂M , depending on the parameter $h \in [0, h_0]$. We denote the symbol of $A = A_h \in \Psi_h^0(\partial M)$ by $a = a(y, \eta, h)$. Thus $a(y, \eta) = a(y, \eta, 0)$ is a smooth function on $T^*\partial M$; we may without loss of generality assume it is compactly supported. We further define states on the algebra $\Psi_h^0(\partial M)$ by

$$\omega_B(A) = \frac{4}{\text{vol}(S^{n-1}) \text{vol}(M)} \int_{B^* \partial M} a(y, \eta) d\mu_B. \quad (4.8)$$

Here, as in Table 1,

$$d\mu_B = \gamma(q) d\sigma \text{ (Dirichlet)}, \quad d\mu_B = \gamma(q)^{-1} d\sigma \text{ (Neumann)}$$

where

$$\gamma(q) = \sqrt{1 - |\eta|^2}, \quad q = (y, \eta).$$

For either Dirichlet or Neumann boundary conditions, the local Weyl law is proved in Lemma 1.2 of [HZ04]:

Proposition 4.1. *Let A_h be a zeroth order semiclassical operator on ∂M . Then,*

$$\lim_{\lambda \rightarrow \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A_{h_j} \varphi_j^b, \varphi_j^b \rangle \rightarrow \omega_B(A) \quad (4.9)$$

Note that in [HZ04] the kernel of A_h was assumed to be disjoint from the singular set of ∂M , but in this article the singular set is empty.

5. KUZNECOV SUM FORMULA FOR THE BOUNDARY INTEGRAL: PROOF OF THEOREM 1.8

The general Kuznecov formula in [Zel92] for C^∞ Riemannian manifolds (M, g) without boundary is a singularity expansion for the distribution

$$S_H(t) = \int_H \int_H E(t, q, q') ds(q') ds(q), \quad (5.1)$$

where $H \subset M$ is a smooth submanifold and where

$$E(t) = \cos t\sqrt{\Delta}$$

is the even wave kernel. The singularities of $S_H(t)$ in the boundaryless case were shown to correspond to trajectories of the geodesic flow which intersect H orthogonally at two distinct times, and to be singular at the difference T of these times. We refer to such trajectories as *H-orthogonal* geodesics.

Theorem 1.8 is a generalization of the Kuznecov formula of [Zel92] to boundary traces on surfaces with concave boundary. We do not consider the full singularity expansion as in [Zel92] but only the singularity at $t = 0$ of

$$\begin{aligned} S_f(t) : &= \int_{\partial M} \int_{\partial M} E_B^b(t, q, q') f(q) f(q') ds(q) ds(q') \\ &= \sum_j \cos t \lambda_j \left| \int_{\partial M} f(q) \varphi_j^b(q) ds(q) \right|^2. \end{aligned} \quad (5.2)$$

where $E_B(t) = \cos t\sqrt{\Delta_B}$ is the even wave kernel with either Dirichlet or Neumann boundary conditions.

Theorem 1.8 is a corollary of Theorems 1, Proposition 2 and Theorem 3 of [HHHZ13], which are proved for general manifolds with boundary.

Theorem 5.1. *Let $\rho \in \mathcal{S}(\mathbb{R})$ be such that $\hat{\rho}$ is identically 1 near 0, and has sufficiently small support. Let $f \in C^\infty(\partial M)$. Then for either the Dirichlet or Neumann boundary conditions,*

$$f(x) = \lim_{\lambda \rightarrow \infty} \frac{\pi}{2} \sum_j \rho(\lambda - \lambda_j) \langle \varphi_j^b, f \rangle \varphi_j^b(x), \quad (5.3)$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\partial M}$ denotes the inner product in $L^2(\partial M)$.

Evidently, Theorem 1.8 follows by taking the inner product with f on both sides of the equation. We now give a relatively self-contained proof, different from that of [HHHZ13], which exploits the concavity of the boundary. For purposes of this article it is only necessary to prove Corollary 1.9. We only use [HHHZ13] to calculate one constant at the end.

5.1. Sketch of the proof. The first step is the following

Lemma 5.2. *There exists $\varepsilon_0 > 0$ so that the*

$$\text{sing supp } S_f(t) \cap (-\varepsilon_0, \varepsilon_0) = \{0\}.$$

Proof. By (4.7) and standard pullback and pushforward calculations for wave front sets as in [Zel92], the singular support of $S_f(t)$ consists of $t = 0$ together with the ‘sojourn times’ equal to lengths of billiard trajectories which hit the boundary orthogonally at both endpoints. Such a billiard trajectory either (i) intersects two distinct components of ∂M , in which case its length is bounded below by the minimum distance d_O between the components, or (ii) intersects the same component orthogonally. However if it starts off orthogonally to the boundary, it cannot intersect the boundary again until it departs from a Fermi normal coordinate chart along the boundary, i.e.

the radius of the maximal tube around each component which is embedded in M . The minimum $\varepsilon(M, g)$ over components of the maximal embedding radius gives a geometric lower bound for its length in this case.

Thus, we may let $\varepsilon_0 = \min\{\varepsilon(M, g), d_{\mathcal{O}}\}$.

□

To prove Theorem 1.8 it thus suffices to determine the singularity at $t = 0$ of $S_f(t)$. Equivalently, we prove a smoothed version and then use a cosine Tauberian theorem. As mentioned in the introduction, we only need a sufficiently accurate asymptotic expansion and remainder to prove Corollary 1.9.

To study the singularity at $t = 0$, we introduce a smooth cutoff $\rho \in \mathcal{S}(\mathbb{R})$ with $\text{supp } \hat{\rho} \subset (-\varepsilon, \varepsilon)$, where $\hat{\rho}$ is the Fourier transform of ρ and $\varepsilon < \varepsilon_0$. With no loss of generality we assume that $\hat{\rho} \in C_0^\infty(\mathbb{R})$ is a positive even function such that $\hat{\rho}$ is identically 1 near 0, has support in $[-1, 1]$ and is decreasing on \mathbb{R}_+ . We then study

$$S_f(\lambda, \rho) = \int_{\mathbb{R}} \hat{\rho}(t) S_f(t) e^{it\lambda} dt. \quad (5.4)$$

Our purpose is to obtain an asymptotic expansion of $S_f(\lambda, \rho)$ as $\lambda \rightarrow \infty$.

Proposition 5.3. *$S_f(\lambda, \rho)$ is a semi-classical Lagrangian distribution whose asymptotic expansion in both the Dirichlet and Neumann cases is given by*

$$S_f(\lambda, \rho) = \frac{\pi}{2} \sum_j (\rho(\lambda - \lambda_j) + \rho(\lambda + \lambda_j)) |\langle \varphi_j^b, f \rangle|^2 = \|f\|_{L^2(\partial M)}^2 + o(1), \quad (5.5)$$

Proof. For $\varepsilon < \varepsilon_0$ in Lemma 5.2, we only need to determine the contribution of the main singularity of $S_f(t)$ (5.2) at $t = 0$. As in [SZ02], the $\rho(\lambda + \lambda_j)$ term contributes $\mathcal{O}(\lambda^{-M})$ for all $M > 0$ and therefore may be neglected.

To show that $S_f(t)$ is a Lagrangian distribution and to determine its singularity at $t = 0$, it suffices to construct a sufficiently precise parametrrix for the non-tangential cutoff $\chi(q, D_t, D_q) E_B^b(t, q, q')$ of the wave kernel for small t in some neighborhood of the diagonal in $\partial M \times \partial M$. In the case of a concave boundary, for small times $E_B^b(t, q, q')$ is singular only when $t = 0$ and $q = q' \in \partial M$. This follows from (4.6) and the fact that there do not exist any broken geodesic billiard trajectories from q to q' for small t except when $t = 0, q = q'$. Thus it suffices to determine the singularity at $t = 0$.

We introduce a pseudo-differential cutoff $\chi(q, D_t, D_q)$ on $\mathbb{R} \times \partial M$ whose symbol vanishes in an arbitrarily small δ -neighborhood of the tangential directions to ∂M . More precisely, as in [HHHZ13], we let $\chi(y, D_t, D_y)$ be a pseudodifferential operator on $\mathbb{R} \times \partial M$ with symbol of the form

$$\chi(y, \tau, \eta) = \zeta(|\eta|_{\tilde{g}}^2 / \tau^2) (1 - \varphi(\eta, \tau)), \quad (5.6)$$

where $\zeta(s)$ is supported where $s \leq 1 - \delta$ for some positive δ , and $\varphi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 near the origin.

We then decompose $E_B^b(t, q, q')$ into an almost tangential part and a part with empty wave front set in tangential directions,

$$E_B^b(t, q, q') = (I - \chi(q, D_t, D_q))E_B^b(t, q, q') + \chi(q, D_t, D_q)E_B^b(t, q, q').$$

We then have corresponding terms in $S_f(t)$,

$$S_f^\varepsilon(t) = \int_{\partial M} \int_{\partial M} (I - \chi(q, D_t, D_q))E_B^b(t, q, q')f(q)f(q')dS(q)dS(q')$$

and

$$S_f^{>\varepsilon}(t) = \int_{\partial M} \int_{\partial M} \chi(q, D_t, D_q)E_B^b(t, q, q')f(q)f(q')dS(q)dS(q').$$

As in [Zel92] (1.6) we express $S_f(t)$ and $S_f(\lambda, \rho)$ in terms of pushforward under the submersion

$$\pi : \mathbb{R} \times \partial M \times \partial M \rightarrow \mathbb{R}, \quad \pi(t, q, q') = t.$$

From (4.6) we find that for $t \in (-\varepsilon, \varepsilon)$,

$$\begin{aligned} WF(S_f^\varepsilon(t)) = \\ \{(0, \tau) : \pi^*(0, \tau) = (0, \tau, 0, 0) \in WF(I - \chi(q, D_t, D_q))E_B^b(t, q, q')\}. \end{aligned}$$

These wave front elements correspond to the points $(0, \tau, \tau\nu_q, \tau\nu_q) \in T_0^*\mathbb{R} \times T_{q, \text{in}}^*M \times T_{q, \text{in}}^*M$, i.e. where both covectors are co-normal to ∂M . Indeed, as in (1.6) of [Zel92] the wave front set of $S_f(t)$ is the set

$$\{(t, \tau) \in T^*\mathbb{R} : \exists(x, \xi, y, \eta) \in C'_t \cap N^*(\partial\Omega) \times N^*\partial\Omega\}$$

in the support of the symbol. However, due to the tangential cutoff $((I - \chi(q, D_t, D_q))E_B^b(t, q, q')$ has no such co-normal vectors in its wave front set. Thus, we may neglect the tangential part of $E_B^b(t, q, q')$ in determining the asymptotics of $S_f(\rho, \lambda)$. But then it follows from [Mel75, Tay76] that the non-tangential part has a geometric optics Fourier integral representation, i.e. $S_f(t)$ is classical co-normal at $t = 0$.

The non-tangential part $\chi(q, D_t, D_q)E_B^b(t, q, q')$ may be expressed in terms of the “free wave kernel” or ambient wave kernel $E_X(t, x, y)$ of (X, g) . Given $q \in \partial M$, we may separate M into an illuminated region (from a source at q) and a shadow region. The hyperplane $T_q\partial M \subset T_qX$ divides the full tangent space into two halfspaces. By concavity, geodesics with initial direction ξ in the lower half-space lie in M for $|t| < \varepsilon$. We call the image of the unit tangent vectors in the lower half space under the geodesic flow up to time ε the ‘illuminated region’ for a point source at q . The complement of the illuminated region in $T_\varepsilon M \cap M$ is the ‘shadow region’. Geodesics with ξ in the upper half plane exponentiate to $X \setminus M$ for at least a short time. If we cutout all ξ whose angle to $T_q\partial M$ is $\leq \delta$, then geodesics in the upper half space remain in $X \setminus M$ for a uniform length of time, which may assume with loss of generality is $> \varepsilon$. We write this set as $T_\delta X$.

We then cut off the ambient cosine wave kernel $E_X(t, x, q)$ in the shadow region, removing the singularities of the ambient kernel due to geodesics that leave M and travel through $X \setminus M$.

Lemma 5.4. *For $|t| < \varepsilon$, the non-tangential part $\chi(q, D_t, D_q)E_B^b(t, q, q')$ of $E_B^b(t, q, q')$ can be expressed as $A(q, D_q)\gamma_q^B\gamma_{q'}^B E_X(t, q, q')\chi(q, D_t, D_q)$ where $E_X(t, x, y)$ is the cosine wave kernel of (X, g) and A is a pseudo-differential operator on ∂M of order zero.*

Indeed, by the wave front calculations of §4.3,

$$E_B^b(t, x, q)\chi(q, D_t, D_q), \quad \text{resp.} \quad E_X^b(t, x, q)\chi(q, D_t, D_q)$$

are Fourier integral operators with the same wave front set equal to a pull-back of the diagonal at $t = 0$, and therefore by the calculus of Fourier integral operators, there exists a pseudo-differential operator A whose composition with the cutoff free wave kernel agrees to any given order with the E_B^b kernel.

Therefore, to prove Proposition 5.3 it is sufficient to consider the integral

$$\begin{aligned} & \int_{\mathbb{R}} \int_{(q, q'): d(q, q') < \varepsilon} \hat{\rho}(t) \chi_q(q, D_t, D_q) e^{it\lambda} \\ & \gamma_B^q \gamma_B^{q'} E_X(t, x, q) f(q) f(q') dS(q) dS(q') dt. \end{aligned} \tag{5.7}$$

We use a Hormander style small time parametrix for $E_X(t, x, q)$, i.e. there exists an amplitude A so that modulo smoothing operators,

$$\gamma_B^b E_X(t, x, q) = \int_{T_q^* X} A(t, x, q, \xi) \exp(i(\langle Exp_q^{-1}(x), \xi \rangle - t|\xi|)) d\xi.$$

The amplitude has order zero. We then take the boundary trace and apply the cutoff operator $\chi(q, D_t, D_q)$, which modifies the amplitude of (5.7) as a sum of terms with the same support as $\chi_q(\tau, \xi)$.

Changing variables $\xi \rightarrow \lambda\xi$, (5.7) may be expressed in the form,

$$\begin{aligned} & \lambda^2 \int_{\mathbb{R}} \int_{\partial M} \int_{\partial M} \int_{T_q^* X} \hat{\rho}(t) e^{it\lambda} \chi_q(\xi) \\ & A(t, q', q, \lambda\xi) \exp(i\lambda(\langle Exp_q^{-1}(q'), \xi \rangle - t|\xi|)) f(q) f(q') d\xi dt dS(q) dS(q'). \end{aligned}$$

We now compute the asymptotics by the stationary phase method.

We already know that for small t , the phase

$$t + \langle Exp_q^{-1}(q'), \xi \rangle - t|\xi|$$

is stationary only at $t = 0$ and $q = q'$. We calculate the expansion by putting the integral over $T_q^* X$ in polar coordinates,

$$\begin{aligned} & \int_{\mathbb{R}} \int_0^\infty \int_{\partial M} \int_{\partial M} \int_{S_q^* X} \hat{\rho}(t) e^{it\lambda} \chi_q(\xi) A(t, q', q) \\ & \exp(i\lambda\rho(\langle Exp_q^{-1}(q'), \omega \rangle - t)) f(q) f(q') \rho^{n-1} d\rho d\omega dt dS(q) dS(q'), \end{aligned}$$

and in these coordinates the phase becomes,

$$\Psi(q, \rho, t, \omega, q') := t + \rho \langle \text{Exp}_q^{-1}(q'), \omega \rangle - t\rho.$$

We get a non-degenerate critical point in the variables (t, ρ) when $\rho = 1, t = \langle \text{Exp}_q^{-1}(q'), \omega \rangle$. Eliminating these variables by stationary phase, we get

$$\frac{1}{\lambda} \int_{\partial M} \int_{\partial M} \int_{S_q^* X} e^{i\lambda \langle \text{Exp}_q^{-1}(q'), \omega \rangle} \chi_q(\omega) \tilde{A}(t, q', q, \rho\omega) f(q) f(q') d\omega dS(q) dS(q'),$$

for another amplitude \tilde{A} . Now the phase is

$$\Psi_q(q', \omega) = \langle \text{Exp}_q^{-1}(q'), \omega \rangle.$$

We fix q and view the phase as a function of $(q', \omega) \in \partial M \times S_{q, \text{in}}^* X$ (the inward pointing unit tangent vectors; due to the cutoff, the amplitude vanishes on the outward pointing vectors). We view $S_{q, \text{in}}^*$ as the lower hemisphere S_-^{n-1} of the sphere $S^{n-1} = S_q^* X$. Note that the integral is compactly supported in the interior of the hemisphere, so critical points on the boundary are irrelevant. We claim that the phase has a critical point if and only if $q' = q$ and $\omega \perp T_q \partial M$, i.e. $\omega = \nu_q$ (ν_q being the inward unit normal). Moreover, the stationary phase point is non-degenerate. Since we are working close to the diagonal, we use geodesic normal coordinates $\text{Exp}_q^{-1}(q') = x$ and consider the inverse image $Y = \text{Exp}_q^{-1}(\partial\Omega)$ of a small piece of $\partial\Omega$ near q in $T_q X$. Let $q(y)$ be a local parametrization of Y with $\partial_j q(y)|_{y=0}$ an orthonormal frame of $T_q \partial\Omega$. Then in any dimension n we may write the phase in local coordinates as

$$\Psi_q(y, \omega) = \langle q(y), \omega \rangle \quad y \in \mathbb{R}^{n-1}, \omega \in S_y^* X.$$

In this parametrization, $q(0) = q$ and Y is locally the graph of a convex function over $T_q \partial\Omega$. As is well-known,

$$\nabla_y \langle q(y), \omega \rangle = \omega^T, \tag{5.8}$$

the tangential projection of ω to $T_{q(y)} \partial\Omega$ and as above we find that $\omega \perp T_q \partial M$ at the stationary phase point. Similarly,

$$\nabla_\omega \langle q(y), \omega \rangle = q(y)^T, \tag{5.9}$$

the tangential projection of $q(y)$ to $T_\omega S^{n-1}$. Then (5.8)-(5.9) occur for $q(y)$ near q if and only if (i) $q(y) = 0$ (in normal coordinates) and $\omega = \nu_q$, or (ii) $\frac{q(y)}{|q(y)|} = \pm \omega = \pm \nu_q$. In the first case, $q' = q$; the second case does not occur for a concave hypersurface. The Hessian in $(y, \omega) \in \partial M \times S_-^{n-1}$ at the critical point has the form,

$$\begin{pmatrix} -II_q & I_q \\ I_q & 0 \end{pmatrix}$$

where II_q is the second fundamental form at q with respect to ν_q (see e.g. [Hör90] §7.7 for the calculation of the upper left block). The lower right

block is zero since $q(y) = 0$ at the critical point. For the off-diagonal blocks, we implicitly identify $T_q\partial\Omega$ with $T_{\nu_q}S_q^*X$ since $\omega = \nu_q$ at the critical point. We parametrize $\omega = \omega(\theta) \in S_-^{n-1}$ so that $\partial_j\omega = e_j$ at ν_q , the same frame we use for $T_q\partial\Omega$. The determinant of the Hessian is evidently of modulus one. In the two dimensional case, we obtain another factor of λ^{-1} from the stationary phase expansion. Therefore (5.7) is asymptotic to a multiple of

$$\int_{\partial M} f^2(q) dS(q),$$

as stated in Proposition 5.5.

To determine the multiple, or more precisely to show that it is positive, we need to find the principal symbol of the pseudo-differential operator in (5.4). In fact, it is a constant equal to 2 when pulled back to ∂M . This follows from the calculations in [HZ12] Proposition 4 and in [HHHZ13], which prove:

Lemma 5.5. *Suppose that $\hat{\rho}$ is supported in $[-\varepsilon, \varepsilon]$ and equal to 1 in a neighbourhood of 0. Then, for sufficiently small ε (depending on δ),*

(1) *the kernels of*

$$\begin{aligned} & \hat{\rho}(t)\chi(y, D_t, D_y) \circ \gamma_q^b \gamma_{q'}^b \cos(t\sqrt{\Delta_D}), \\ & \hat{\rho}(t)\gamma_q^b \gamma_{q'}^b \cos(t\sqrt{\Delta_D}) \circ \chi(y, D_t, D_y) \end{aligned}$$

are distributions conormal to $\{y = y', t = 0\}$ with principal symbol

$$2\chi(y, \tau, \eta) \left(1 - \frac{|\eta|_{\tilde{g}}^2}{\tau^2}\right)^{\frac{1}{2}}; \quad (5.10)$$

(2) *the kernels of*

$$\begin{aligned} & \hat{\rho}(t)\chi(y, D_t, D_y) \circ \gamma_q^b \gamma_{q'}^b \cos(t\sqrt{\Delta_N}), \\ & \hat{\rho}(t)\gamma_q^b \gamma_{q'}^b \cos(t\sqrt{\Delta_N}) \circ \chi(y, D_t, D_y) \end{aligned}$$

are distributions conormal to $\{y = y', t = 0\}$ with principal symbol

$$2\chi(y, \tau, \eta) \left(1 - \frac{|\eta|_{\tilde{g}}^2}{\tau^2}\right)^{-\frac{1}{2}}. \quad (5.11)$$

This completes the proof of Theorem 1.8. \square

Remark:

Above, we use the notation $\cos(t\sqrt{\Delta_D})$ resp. $\cos(t\sqrt{\Delta_N})$ in place of $E_B(t)$ since the formula is different in the Dirichlet, resp. Neumann cases. Also, the symbols are homogeneous analogues of (3.1).

6. PROOF OF THEOREM 1.3

In this section, we give a proof of Proposition 1.7 and Theorem 1.3 for Neumann eigenfunctions. The argument for Dirichlet eigenfunctions is exactly the same.

6.1. Proof of Proposition 1.7. Firstly let $\beta \subset \partial M$ be an interval and let $f \in C_0^\infty(\partial M)$ be a function such that

$$\begin{aligned} f(x) &\geq 0 & x \in \partial M \\ f(x) &= 0 & x \notin \beta \\ f(x) &> 0 & x \in \beta \end{aligned}$$

Denote by $N(\lambda)$ the number of eigenfunctions in $\{j \mid \lambda < \lambda_j < 2\lambda\}$. We have by Theorem 1.8 and Chebyshev's inequality,

$$\frac{1}{N(\lambda)} \left| \left\{ j \mid \lambda < \lambda_j < 2\lambda, \left| \int_{\gamma_i} f \varphi_j ds \right|^2 \geq \lambda_j^{-1} M \right\} \right| = O_f\left(\frac{1}{M}\right).$$

Corollary 1.9 follows immediately.

Note that

$$\int_{\partial M} f |\varphi_j|^2 ds \leq \int_{\partial M} f |\varphi_j| ds \sup_{x \in \partial M} |\varphi_j(x)|.$$

For a density 1 subsequence $\{\varphi_j\}_{j \in A}$ which satisfies (1.4), we have

$$\int_{\partial M} f |\varphi_j|^2 ds \gg_f 1.$$

Therefore from the third assumption in Theorem 1.1,

$$\int_{\partial M} f |\varphi_j| ds > 2M \lambda_j^{-\frac{1}{2}}$$

is satisfied for all sufficiently large $j \in A$. Combining with Corollary 1.9, this proves the existence of a subsequence of density $\geq 1 - \frac{c}{M}$ which satisfies

$$\int_{\partial M} f |\varphi_j| ds > \left| \int_{\partial M} f \varphi_j ds \right|.$$

Putting

$$A_\beta = \left\{ j : \int_{\partial M} f |\varphi_j| ds > \left| \int_{\partial M} f \varphi_j ds \right| \right\},$$

we obtain

$$\liminf_{N \rightarrow \infty} \frac{1}{N} |\{j < N : j \in A_\beta\}| \geq 1 - \frac{c}{M}.$$

Since the left quantity does not depend on M , this proves Proposition 1.7 for Neumann eigenfunctions.

6.2. Proof of Theorem 1.3. Note that because f is positive on β , a function φ_j has a sign change on β if and only if $j \in A_\beta$. Let $R \in \mathbb{N}$ be fixed, and let $\beta_1, \dots, \beta_R \subset \partial M$ be disjoint segments in ∂M . Then by Proposition 1.7, each A_{β_k} ($1 \leq k \leq R$) is a natural density 1 subset of \mathbb{N} . Therefore $A(R) = \cap_{k=1}^R A_{\beta_k}$ is a density 1 subset of \mathbb{N} , and any φ_j with $j \in A(R)$ has at least R sign changes along ∂M . We apply the following lemma to conclude Theorem 1.6 for Neumann eigenfunctions.

Lemma 6.1. *Let a_n be a sequence of real numbers such that for any fixed $R > 0$, $a_n > R$ is satisfied for almost all n . Then there exists a density 1 subsequence $\{a_n\}_{n \in A}$ such that*

$$\lim_{\substack{n \rightarrow \infty \\ n \in A}} a_n = +\infty.$$

Proof. Let n_k be the least number such that for any $n \geq n_k$,

$$\frac{1}{n} |\{j \leq n \mid a_j > k\}| > 1 - \frac{1}{2^k}.$$

Note that n_k is nondecreasing, and $\lim_{k \rightarrow \infty} n_k = +\infty$.

Define $A_k \subset \mathbb{N}$ by

$$A_k = \{n_k \leq j < n_{k+1} \mid a_j > k\}.$$

Then for any $n_k \leq m < n_{k+1}$,

$$\{j \leq m \mid a_j > k\} \subset \bigcup_{l=1}^k A_l \cap [1, m];$$

which implies by the choice of n_k that

$$\frac{1}{m} \left| \bigcup_{l=1}^k A_l \cap [1, m] \right| > 1 - \frac{1}{2^k}.$$

This proves

$$A = \bigcup_{k=1}^{\infty} A_k$$

is a density 1 subset of \mathbb{N} , and by the construction we have

$$\lim_{\substack{n \rightarrow \infty \\ n \in A}} a_n = +\infty.$$

□

6.3. Topological arguments: Euler inequality. As done in [JZ13], we can give a graph structure (i.e. the structure of a one-dimensional CW complex) to Z_{φ_λ} as follows.

- (1) For each embedded circle which does not intersect γ , we add a vertex.
- (2) Each singular point is a vertex.
- (3) Each intersection point in $\partial M \cap \left(\overline{Z_{\varphi_\lambda} \setminus \partial M} \right)$ is a vertex.
- (4) Edges are the arcs of $Z_{\varphi_\lambda} \cup \partial M$ which join the vertices listed above.

This way, we obtain a graph embedded into the surface M . We recall that an embedded graph G in a surface M is a finite set $V(G)$ of vertices and a finite set $E(G)$ of edges which are simple (non-self-intersecting) curves in M such that any two distinct edges have at most one endpoint and no interior points in common. The *faces* f of G are the connected components of $M \setminus V(G) \cup \bigcup_{e \in E(G)} e$. The set of faces is denoted $F(G)$. An edge $e \in E(G)$ is *incident* to f if the boundary of f contains an interior point of e . Every edge is incident to at least one and to at most two faces; if e is incident to f then $e \subset \partial f$. The faces are not assumed to be cells and the sets $V(G), E(G), F(G)$ are not assumed to form a CW complex. Indeed the faces of the nodal graph of eigenfunctions are nodal domains, which do not have to be simply connected.

Remark 6.2. *Every vertex has degree ≥ 2 , since any interior singular point is locally an intersection of simple curves, as follows from the local Bers expansion around a zero [Cha73].*

Let $\iota : M \hookrightarrow \tilde{M}$ be an embedding into a closed surface. We assume that $\tilde{M} \setminus \iota(M)$ is a disjoint union of disks, and we denote by h_M the number of connected components of $\tilde{M} \setminus \iota(M)$. (Such pair ι, \tilde{M} can be constructed, for example, by mapping cone.)

Let $v(\varphi_\lambda)$ (resp. $\tilde{v}(\varphi_\lambda)$) be the number of vertices, $e(\varphi_\lambda)$ (resp. $\tilde{e}(\varphi_\lambda)$) be the number of edges, $f(\varphi_\lambda)$ (resp. $\tilde{f}(\varphi_\lambda)$) be the number of faces, and $m(\varphi_\lambda)$ (resp. $\tilde{m}(\varphi_\lambda)$) be the number of connected components of the graph G inside M (resp. $\iota(G)$ inside \tilde{M}).

Then we have

$$\begin{aligned} v(\varphi_\lambda) &= \tilde{v}(\varphi_\lambda) \\ e(\varphi_\lambda) &= \tilde{e}(\varphi_\lambda) \\ m(\varphi_\lambda) &= \tilde{m}(\varphi_\lambda) \end{aligned}$$

while

$$f(\varphi_\lambda) + h_M = \tilde{f}(\varphi_\lambda).$$

Now by Euler's formula (Appendix F, [Gro12]),

$$v(\varphi_\lambda) - e(\varphi_\lambda) + f(\varphi_\lambda) - m(\varphi_\lambda) + h_M \tag{6.1}$$

$$= \tilde{v}(\varphi_\lambda) - \tilde{e}(\varphi_\lambda) + \tilde{f}(\varphi_\lambda) - \tilde{m}(\varphi_\lambda) \geq 1 - 2g_{\tilde{M}} \tag{6.2}$$

where $g_{\tilde{M}}$ is the genus of the surface \tilde{M} .

Theorem 6.3. *Let*

$$n(\varphi_j) = \begin{cases} \#Z_{\varphi_j} \cap \partial M & (\text{Neumann case}) \\ \#\Sigma_{\varphi_j} \cap \partial M & (\text{Dirichlet case}) \end{cases}$$

Then we have:

$$N(\varphi_j) \geq \frac{1}{2}n(\varphi_j) + 2 - 2g_{\tilde{M}} - h_M.$$

Proof. Since faces of G on M are nodal domains of φ_j , $f(\varphi_j) = N(\varphi_j)$. Observe that, in Neumann case, points in $Z_{\varphi_j} \cap \partial M$ ($\Sigma_{\varphi_j} \cap \partial M$, in Dirichlet case) correspond to vertices having degree at least 3 on the graph. Also, every vertex has degree ≥ 2 (Remark 6.2). Therefore,

$$\begin{aligned} 0 &= \sum_{x: \text{vertices}} \deg(x) - 2e(\varphi_j) \\ &\geq 2(v(\varphi_j) - n(\varphi_j)) + 3n(\varphi_j) - 2e(\varphi_j), \end{aligned}$$

i.e.

$$e(\varphi_j) - v(\varphi_j) \geq \frac{1}{2}n(\varphi_j).$$

Plugging into (6.1) with $m(\varphi_j) \geq 1$, we obtain

$$N(\varphi_j) \geq \frac{1}{2}n(\varphi_j) + 2 - 2g_{\tilde{M}} - h_M.$$

□

Theorem 1.3 is an immediate consequence of Theorem 1.6 and the topological argument, Theorem 6.3.

APPENDIX A. APPENDIX ON DENSITY ONE

Define the natural density of a set $A \in \mathbb{N}$ by

$$\lim_{X \rightarrow \infty} \frac{1}{X} |\{x \in A \mid x < X\}|$$

whenever the limit exists. We say “almost all” when corresponding set $A \in \mathbb{N}$ has the natural density 1. Note that intersection of finitely many density 1 set is a density 1 set. When the limit does not exist we refer to the lim sup as the upper density and the lim inf as the lower density.

A.1. Diagonal argument. Let $\{f_n\} \subset C^\infty(H)$ be a countable dense subset of $C^0(H)$ with respect to the sup norm. For each n , we have a family $\Lambda_n(\lambda)$ of subsets for which

$$\frac{1}{N(\lambda)} \# \Lambda_n(\lambda) \rightarrow 1$$

and such that

$$\begin{cases} \int_H f \varphi_j^2 ds \rightarrow \int_H f d\nu, \text{ as } \lambda \rightarrow \infty \text{ with } \lambda_j \in \Lambda_n(\lambda) \\ \lambda_j^{-1/2} \int_H f \varphi_j ds \rightarrow 0. \end{cases} \quad (\text{A.1})$$

We may assume that $\Lambda_{n+1}(\lambda) \subset \Lambda_n(\lambda)$. For each n let Λ_n be large enough so that

$$\frac{1}{N(\lambda)} \# \Lambda_n(\lambda) \geq 1 - \frac{1}{n}, \quad \lambda \geq \Lambda_n.$$

Define

$$\Lambda_\infty(\lambda) : \Lambda_n(\lambda), \quad \Lambda_k \leq \lambda \leq \Lambda_{k+1}.$$

Then

$$\frac{1}{N(\lambda)} \# \Lambda_\infty(\lambda) \geq 1 - \frac{1}{n}, \quad \lambda \geq \Lambda_n,$$

so $D^*(\Lambda_\infty) = D_*(\Lambda_\infty) = 1$ and (A.1) is valid for the sequence Λ_∞ .

REFERENCES

- [Bia13] M. Bialy. Hopf rigidity for convex billiards on the hemisphere and hyperbolic plane. *Discrete Contin. Dyn. Syst.*, 33(9):3903–3913, 2013.
- [BSC90] L. A. Bunimovich, Ya. G. Sinai, and N. I. Chernov. Markov partitions for two-dimensional hyperbolic billiards. *Uspekhi Mat. Nauk*, 45(3(273)):97–134, 221, 1990.
- [Bur05] N. Burq. Quantum ergodicity of boundary values of eigenfunctions: a control theory approach. *Canad. Math. Bull.*, 48(1):3–15, 2005.
- [CFS82] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.
- [Cha73] J. Chazarain. Construction de la paramétrix du problème mixte hyperbolique pour l'équation des ondes. *C. R. Acad. Sci. Paris Sér. A-B*, 276:A1213–A1215, 1973.
- [CS87] N.I. Chernov and Ya. G. Sinai. *Ergodic properties of some systems of two-dimensional disks and three-dimensional balls*, volume 42. 1987.
- [CTZ13] H. Christianson, J. A. Toth, and S. Zelditch. Quantum ergodic restriction for Cauchy data: interior que and restricted que. *Math. Res. Lett.*, 20(3):465–475, 2013.
- [Gro12] M. Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. *J. ACM*, 59(5):Art. 27, 64, 2012.
- [GRS13] A. Ghosh, A. Reznikov, and P. Sarnak. Nodal Domains of Maass Forms I. *Geom. Funct. Anal.*, 23(5):1515–1568, 2013.
- [HHHZ13] X. Han, A. Hassell, H. Hezari, and S. Zelditch. Completeness of boundary traces of eigenfunctions. *arXiv:1311.0935*, 2013.
- [Hör90] L. Hörmander. *The analysis of linear partial differential operators. I-IV*. Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.
- [HZ04] A. Hassell and S. Zelditch. Quantum ergodicity of boundary values of eigenfunctions. *Comm. Math. Phys.*, 248(1):119–168, 2004.
- [HZ12] H. Hezari and S. Zelditch. C^∞ spectral rigidity of the ellipse. *Anal. PDE*, 5(5):1105–1132, 2012.
- [JZ13] J. Jung and S. Zelditch. Number of nodal domains and singular points of eigenfunctions of negatively curved surfaces with an isometric involution. *arXiv:1310.2919*, 2013.
- [KSS89] A. Krámli, N. Simányi, and D. Szász. Dispersing billiards without focal points on surfaces are ergodic. *Comm. Math. Phys.*, 125(3):439–457, 1989.
- [Mel75] R. B Melrose. Microlocal parametrices for diffractive boundary value problems. *Duke Math. J.*, 42(4):605–635, 1975.
- [MS78] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978.
- [MT85] R. B. Melrose and M. E. Taylor. Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. *Adv. in Math.*, 55(3):242–315, 1985.
- [Sin70] Ya. G. Sinai. Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. *Uspekhi Mat. Nauk*, 25(2 (152)):141–192, 1970.

- [Sto89] L. Stojanov. An estimate from above of the number of periodic orbits for semi-dispersed billiards. *Comm. Math. Phys.*, 124(2):217–227, 1989.
- [Sto99] L. Stojanov. Exponential instability for a class of dispersing billiards. *Ergodic Theory Dynam. Systems*, 19(1):201–226, 1999.
- [SZ02] C. D. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 114(3):387–437, 2002.
- [SZ13] C. D. Sogge and S. Zelditch. Focal points and sup-norms of eigenfunctions. *arXiv:1311.3999*, 2013.
- [SZ14] C. D. Sogge and S. Zelditch. Sup norms of cauchy data of eigenfunctions on manifolds with concave boundary. *arXiv:1411.1035*, 2014.
- [Tay76] M. E. Taylor. Grazing rays and reflection of singularities of solutions to wave equations. *Comm. Pure Appl. Math.*, 29(1):1–38, 1976.
- [TZ09] J. A. Toth and S. Zelditch. Counting nodal lines which touch the boundary of an analytic domain. *J. Differential Geom.*, 81(3):649–686, 2009.
- [Woj94] M. Wojtkowski. Two applications of jacobi fields to the billiard ball problem. *J. Differential Geom.*, 40(1):155–164, 1994.
- [Zel92] S. Zelditch. Kuznecov sum formulae and Szegő limit formulae on manifolds. *Comm. Partial Differential Equations*, 17(1-2):221–260, 1992.
- [ZL07] H. Zhang and J. Lian. Hyperbolic behavior of Jacobi fields along billiard flows. In *Proceedings of the 4th International Conference on Impulsive and Hybrid Dynamical Systems*, volume 54 of *Proc. Sympos. Pure Math.*, pages 1794–1798. Watan Press, Providence, RI, 2007.

DEPARTMENT OF MATHEMATICAL SCIENCE, KAIST, DAEJEON 305-701, SOUTH KOREA

Current address: School of Mathematics, IAS, Princeton, NJ 08540, USA

E-mail address: `junehyuk@ias.edu`

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, USA

E-mail address: `zelditch@math.northwestern.edu`